Quantum Mechanics I Midterm Exam

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This midterm consists of six problems, for a total of 100 points. I have arranged the problems in order of increasing difficulty. Make sure that you attempt to answer all the questions and all the problems. If you get stuck with a question for too long, proceed to the next one. Please answer as clearly, precisely and thoroughly as possible. May the muses be with you.

Problem 1 (20 points)

Imagine that two operators $\hat{A}$ and $\hat{B}$ satisfy the commutation relations $[\hat{A}, \hat{B}] = i\lambda$, where λ is a constant.

1. What inequality is satisfied by $\Delta A \cdot \Delta B$?
2. How is ΔA defined?

Problem 2 (10 points)

Show that $(|\varphi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\varphi|$. Please make sure than you justify every step in your proof.

Problem 3 (10 points)

1. What is the difference between a pure and a mixed state? Answer the question both formally and physically.
2. Show that under a unitary time evolution pure states remain pure.

Problem 4 (20 points)

Let the propagator $U(t, t_0)$ be the operator defined by the relation $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$.

1. Is $U(t, t_0)$ an observable?
2. Express $U(t, t_0)$ in terms of the Hamiltonian $\hat{H}$.
3. Under what condition is your answer valid?
4. Show that $|\psi(t)\rangle$ as defined above satisfies the Schrödinger equation. If you do not know what U is, simply write down Schrödinger's equation.

Please turn!

Problem 5 (20 points)

Let $|a\rangle$ and $|b\rangle$ and $|c\rangle$ be normalized eigenvectors of the observable $\hat{M}$,

$$\hat{M}|a\rangle = a|a\rangle, \quad \hat{M}|b\rangle = b|b\rangle, \quad \hat{M}|c\rangle = c|c\rangle, \tag{1}$$

where a, b and c are pairwise distinct. The states $|a\rangle$ and $|b\rangle$ are eigenvectors of the Hamiltonian to the *same* eigenvalue E_{ab}, whereas the state $|c\rangle$ is an eigenvector of the Hamiltonian with eigenvalue E_c. At time t_0 the system is prepared in a state

$$|\psi(t_0)\rangle = \alpha|a\rangle + \beta|b\rangle + \gamma|c\rangle. \tag{2}$$

At time t_1 we measure the energy of the system, and at time t_2 we measure the energy of the system again.

1. What condition has to be satisfied by the coefficients α, β and γ?
2. What is the state of the system immediately before and after the measurement at t_1?
3. What is the probability of measuring an energy E_{ab} at time t_2?
4. What is the expectation value of the energy at time t_2?

Problem 6 (20 points)

Let $\hat{x}$ and $\hat{p}$ be two observables satisfying $[\hat{x}, \hat{p}] = i\hbar$, and let a be an arbitrary real number.

1. What does the operator $\exp\left(\frac{i}{\hbar}\hat{p}\,a\right)\hat{x}\exp\left(-\frac{i}{\hbar}\hat{p}\,a\right)$ equal to? If you do not know the answer proceed to the next step and consider the hint.
2. Prove your answer. *Hint:* Consider matrix elements of the operator between two arbitrary sates $|\varphi\rangle$ and $|\psi\rangle$.