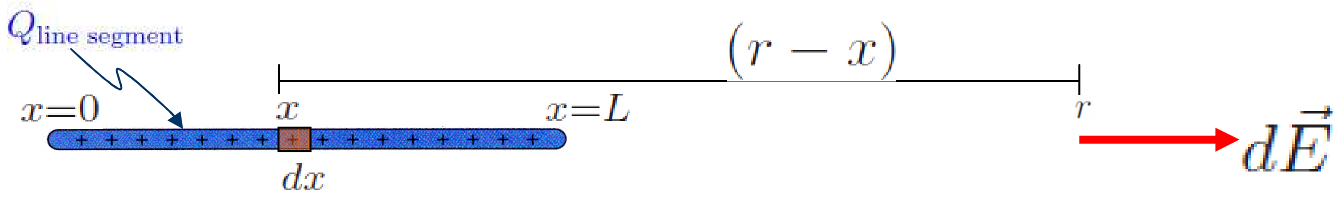
THE LINE SEGMENT

for positions along
the axis of the line segment

$$\vec{E}_{\substack{\text{due to}\\ \text{line segment}\\ \text{of length } L}} = \int_{\substack{\text{all tiny segments}\\ \text{forming the line segment}}} \left(d\vec{E}_{\substack{\text{due to}\\ \text{tiny segment}}} \right)$$



Since all contributions to $d\vec{E}$ are parallel, we can work with components.

$$dE_{\substack{\text{due to}\\ \text{tiny segment}\\ \text{of length } dx}} = k\frac{\left(\substack{\text{charge of}\\ \text{tiny segment}}\right)}{\left(\substack{\text{distance to}\\ \text{tiny segment}}\right)^2} = k\frac{\left(Q_{\text{line segment}}\frac{dx}{L}\right)}{(r-x)^2}$$

$$= k\frac{(\lambda\, dx)}{(r-x)^2}$$

Now do MATH.

$$E_{\substack{\text{due to}\\ \text{line segment}\\ \text{of length } L}} = \int_{x=0}^{x=L} k\frac{(\lambda\, dx)}{(r-x)^2}$$

Let $x' = x - r$

$x = x' + r$ and $dx = dx'$

$$\overset{M}{=} \int_{x'=-r}^{x'=L-r} k\frac{(\lambda\, dx')}{(x')^2}$$

(-1)^2=1

$$\overset{M}{=} k\lambda \int_{x'=-r}^{x'=L-r} \frac{(dx')}{(x')^2}$$

$$\int du\; u^n = \frac{u^{n+1}}{n+1}$$

$$\int \frac{du}{u^2} = \int du\; u^{-2} = \frac{u^{-1}}{-1}$$

$$\overset{M}{=} k\lambda \left(\frac{1}{-x'}\right]_{-r}^{L-r}$$

$$\overset{M}{=} -k\lambda\left(\frac{1}{L-r} - \frac{1}{-r}\right)$$

$$= -k\lambda\left(\frac{-L}{r(r-L)}\right)$$

$$= k\lambda\left(\frac{L}{r(r-L)}\right)$$

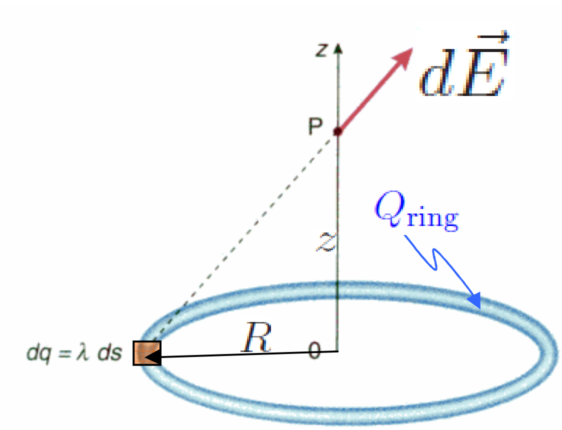
$$= k\left(\frac{Q_{\text{line segment}}}{r(r-L)}\right)$$

for positions r along
the axis of the line segment

THE RING

for positions along the axis of the ring

$$\vec{E}_{\substack{\text{due to}\\\text{ring of}\\\text{radius } R}} = \int_{\substack{\text{all tiny segments}\\\text{forming the ring}}} \left(d\vec{E}_{\substack{\text{due to}\\\text{tiny segment}}} \right)$$



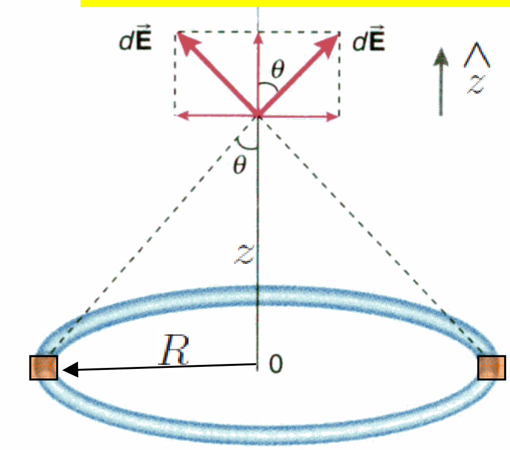
Since all contributions to $d\vec{E}$ are NOT parallel, we MUST work with VECTORS.

$$d\vec{E}_{\substack{\text{due to}\\\text{tiny segment}\\\text{of length } ds}} = k \frac{\left(\substack{\text{charge of}\\\text{tiny segment}}\right)}{\left(\substack{\text{distance to}\\\text{tiny segment}}\right)^2} \widehat{\left(\substack{\text{unit vector}\\\text{pointing away from}\\\text{tiny segment}}\right)}$$

$$= k \frac{\left(Q_{\text{ring}} \frac{ds}{2\pi R}\right)}{(R^2 + z^2)} \hat{r}_{\substack{\text{pointing away from}\\\text{tiny segment}}}$$

$$= k \frac{(\lambda\, ds)}{(R^2 + z^2)} \hat{r}_{\substack{\text{pointing away from}\\\text{tiny segment}}}$$

Now do MATH.



$$\stackrel{M}{=} k \frac{(\lambda\, ds)}{(R^2 + z^2)} \Big(\underbrace{(\cos\theta)\hat{z}}_{\textit{vertical}} + \underbrace{(-\sin\theta)\hat{R}}_{\textit{horizontal}} \Big)$$

$$d\vec{E}_{\substack{\text{vertical,}\\\text{due to}\\\text{tiny segment}\\\text{of length } ds}} \stackrel{M}{=} k \frac{(\lambda\, ds)}{(R^2 + z^2)} \Bigg(\underbrace{\left(\frac{z}{\sqrt{R^2 + z^2}}\right)\hat{z}}_{\textit{vertical}} \Bigg)$$

$$\stackrel{M}{=} \underbrace{\left(k\lambda \frac{z}{(R^2 + z^2)^{3/2}} \right)}_{\text{same constant for each segment}} ds\, \hat{z}$$

$$\vec{E}_{\substack{\text{due to}\\\text{ring of}\\\text{radius } R}} \stackrel{M}{=} \int_{s=0}^{s=2\pi R} \left(k\lambda \frac{z}{(R^2 + z^2)^{3/2}} \right) ds\, \hat{z}$$

$$\stackrel{M}{=} \left(k\lambda \frac{z}{(R^2 + z^2)^{3/2}} \right) \hat{z} \int_{s=0}^{s=2\pi R} ds$$

$$\stackrel{M}{=} \left(k\lambda \frac{z}{(R^2 + z^2)^{3/2}} \right) \hat{z}\, (2\pi R)$$

$$\stackrel{M}{=} k \frac{Q_{\text{ring}}\, z}{(R^2 + z^2)^{3/2}} \hat{z} = k \frac{\left(\substack{\text{charge of}\\\text{ring}}\right)\left(\substack{\text{height}\\\text{above}}\right)}{\left(\substack{\text{distance to}\\\text{point on this ring}}\right)^3} \widehat{\left(\substack{\text{unit vector}\\\text{pointing}\\\text{along the axis}}\right)}$$

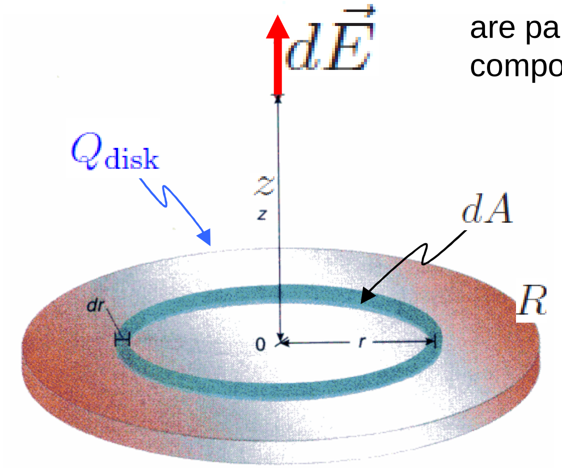
for positions z along the axis of the ring

THE DISK (from many rings)

for positions along the axis of the disk

$$\vec{E}_{\substack{\text{due to}\\ \text{disk of}\\ \text{radius } R}} = \int_{\substack{\text{all tiny rings}\\ \text{forming the disk}}} \left(d\vec{E}_{\substack{\text{due to}\\ \text{tiny rings}}} \right)$$

Since all contributions to $d\vec{E}$ are parallel, we can work with components.



$$dE_{\substack{\text{due to}\\ \text{tiny ring of radius } r}} = k\frac{\left(\begin{array}{c}\text{charge of}\\ \text{tiny ring}\end{array}\right)\left(\begin{array}{c}\text{height}\\ \text{above}\end{array}\right)}{\left(\begin{array}{c}\text{distance to}\\ \text{point on this ring}\end{array}\right)^3}$$

$$= k\frac{\left(Q_{\text{disk}}\frac{dA}{\pi R^2}\right) z}{(r^2+z^2)^{3/2}}$$

$$\stackrel{M}{=} k\frac{(\sigma\ dA)\ z}{(r^2+z^2)^{3/2}}$$

Now do MATH.

$$\stackrel{M}{=} k\frac{(\sigma\ 2\pi r\ dr)\ z}{(r^2+z^2)^{3/2}}$$

$$\stackrel{M}{=} k\ 2\pi\sigma z\ \frac{r\ dr}{(r^2+z^2)^{3/2}}$$

$$E_{\substack{\text{due to}\\ \text{disk of}\\ \text{radius } R}} \stackrel{M}{=} \int_{r=0}^{r=R} k\ 2\pi\sigma z\ \frac{r\ dr}{(r^2+z^2)^{3/2}}$$

$$\stackrel{M}{=} k\ 2\pi\sigma z \int_{r=0}^{r=R} \frac{r\ dr}{(r^2+z^2)^{3/2}}$$

Let $u = (r^2+z^2)$. Then $du = 2r\ dr$.

$$\stackrel{M}{=} k\ 2\pi\sigma z \int_{u=z^2}^{u=R^2+z^2} \frac{\frac{1}{2}du}{(u)^{3/2}}$$

$$\int du\ u^n = \frac{u^{n+1}}{n+1}$$

$$\stackrel{M}{=} k\ 2\pi\sigma z\ \ \frac{1}{2}\left(\frac{1}{-\frac{1}{2}u^{1/2}}\right]_{u=z^2}^{u=R^2+z^2}$$

$$\stackrel{M}{=} k\ 2\pi\sigma z\ \ (-1)\left(\frac{1}{\sqrt{R^2+z^2}} - \frac{1}{z}\right)$$

$$\stackrel{M}{=} k\ 2\pi\sigma z\ \left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right)$$

$$\vec{E}_{\substack{\text{due to}\\ \text{disk of}\\ \text{radius } R}} \stackrel{M}{=} k\ 2\pi\sigma\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right)\left(\widehat{\begin{array}{c}\text{unit vector}\\ \text{pointing away,}\\ \text{perpendicular to the disk}\end{array}}\right)$$

for positions z along the axis of the disk

THE PLANE (from a large disk)

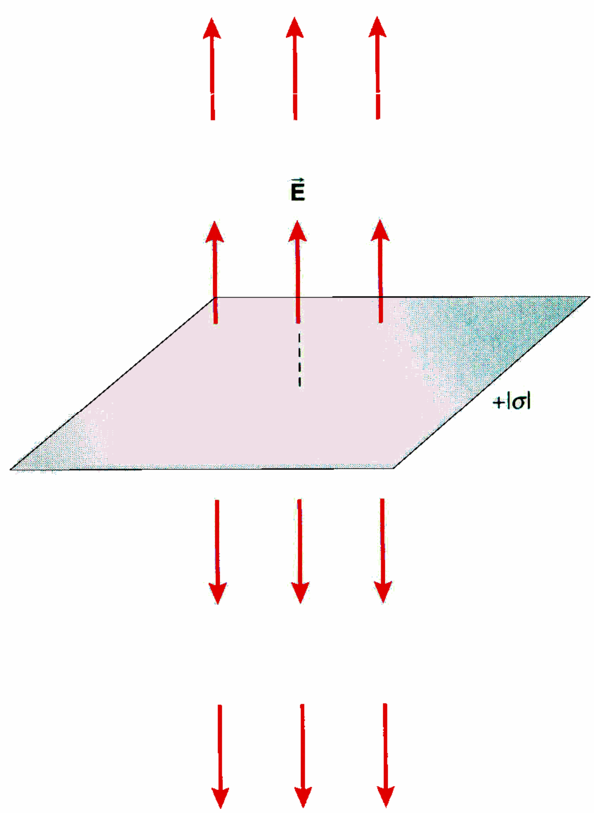
$$\vec{E}_{\substack{\text{due to}\\ \text{an infinite plane}}} = \lim_{R\to\infty} \vec{E}_{\substack{\text{due to}\\ \text{disk of}\\ \text{radius } R}}$$

Now do MATH.

$$\stackrel{M}{=} \lim_{R\to\infty} \left(k\ 2\pi\sigma \left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) \widehat{\begin{pmatrix} \text{pointing away,} \\ \text{perpendicular} \end{pmatrix}} \right)$$

$$\stackrel{M}{=} k\ 2\pi\sigma \widehat{\begin{pmatrix} \text{pointing away,} \\ \text{perpendicular} \end{pmatrix}}$$

$$\stackrel{M}{=} \frac{\sigma}{2\epsilon_0} \widehat{\begin{pmatrix} \text{pointing away,} \\ \text{perpendicular} \end{pmatrix}}$$



TWO OPPOSITELY CHARGED PLANES (superposition)