NOTES ON THE LOGISTIC MAP

MAPS

In science, we use numbers to describe the state of a *system*, such as '10,132 zebra fish are in the lake' or 'the asteroid is moving at 10 kilometers per second and is 32,000 km away from this point.' Given those numbers and a model or theory for how the system changes in time, we can predict the future. If we consider making predictions at regularly spaced intervals or whenever specific conditions are met, we might represent our knowledge as a *map*. The map is the function that predicts the observations at a specific time in the future, based upon the current observations.

Examples:

Clock. Based upon experience, we note that every 3 hours, the little hand on an analog clock advances in angle by a right angle, 90?. So, if we measure the angle in degrees (with respect to the up direction), the map can be stated in English as:

> "if the angle of the little hand is a, then 3 hours later, the measured angle is a+90 (in clockwise degrees.)"

The shorthand for this might be:

"a ? a + 90, every 3 hours".

Slowing down of a rolling ball. Again, based upon experience, we might find that a rolling ball on a linoleum floor slows down to ½ of its speed every time it crosses a tile. If we measure the speed s in meters per second (m/s), we might write this as:

"s ? $s/2$, each tile."

Repeating maps:

These maps can be repeated (also called *iterated*) to predict further into the future. You can check, in these examples, that a ? a + 180 every 6 hours and that s ? s/8, every time the rolling ball crosses three tiles.

We will use maps stated in this format to simulate how systems evolve in time.

What we will find is that amazingly intricate behavior can arise from very simple maps.

LOGISTIC MAP

We will spend a little time in this class studying the famous “logistic map” and its relatives.

Much of this discussion was gone over quickly in class on Sept. 12 and it overlaps very much with the discussion in the book *Chaos* by Gleick.

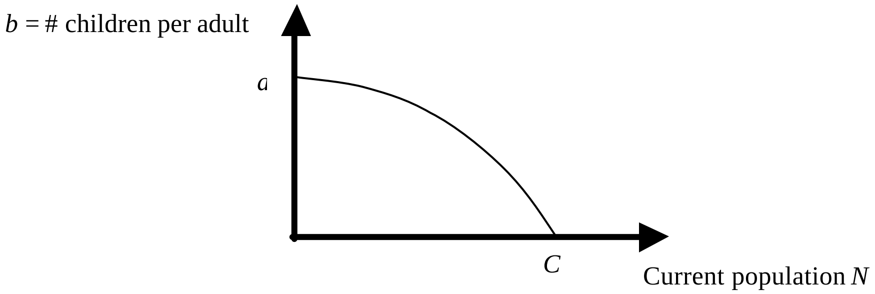
Let’s see how the logistic map arises from a simplified analysis of population growth in a limited environment. This analysis will assume a reproductive rate that depends on the available resources: when the population is high, the available resources are reduced, so that the reproductive success is reduced. Let us assume that the animals reproduce to give the next generation, then die off before the next generation reproduces.

Let the number of animals at any given generation be N (which is really an integer, but we will allow to be any value.) If there are no restrictions due to reduced resources, the number of animals in the next generation is assumed to be proportional to N – there is a fixed number of children per adult. We can write this as

$$N \;?\;\; a\,N \quad \text{(when } N \text{ is small.)} \qquad \text{[Equation 1]}$$

This just says that the number of births is proportional to the total population.

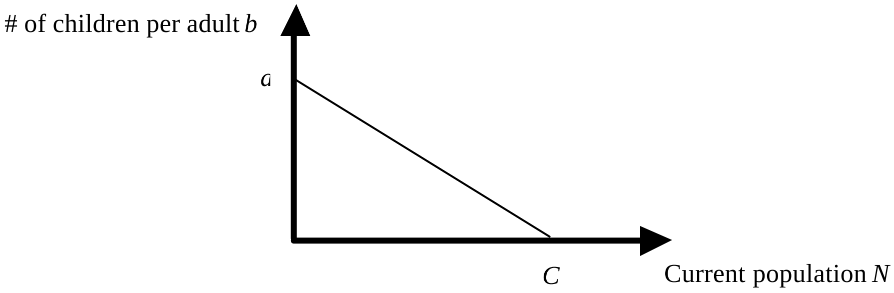
Now, lets say that reproductive success declines from its maximum a with population increase, due to reduced resources. The higher N is, the fewer offspring reproduced. Let C be the maximum capacity of the environment. We might see a reproductive success curve that looks something like this:



The important feature of the reproductive success curve is that the reproductive success goes to zero as N increases to the capacity C. If you know the function $b(N)$, the average births per adult in an environment witn N adults, you can find the size of the next generation by using the map

$$N \;?\;\; b(N)\,N \qquad \text{[Equation 2]}$$

It turns out the general properties of this map are not sensitive to the details of the function $b(N)$. So we can assume a simple form for how the number of children per adult depends on the current population. The simplest form that we can assume is a straight line, so let’s assume that, as plotted on the next page:



The equation of this line is $b = a\,(1 - N/C)$. (Confirm that this gives $b=a$ when $N=0$ and $b=0$ when $N=C$.) By replacing $b(N)$ with $a\,(1 - N/C)$ in Equation 2, we get the map

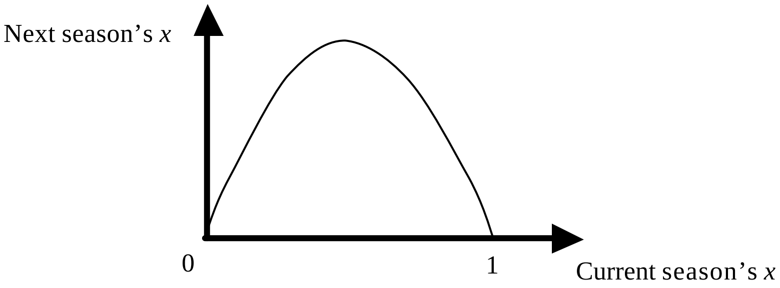
$$N \ ? \ \ a\,(1 - N/C)\,N \qquad \text{[Equation 3]}$$

Finally, we define the fraction of capacity by $x = N/C$. When $x=0$, the population is small and when x becomes close to 1, the population is approaching full capacity. So replacing N with N/C in Equation 3, we get the <u>logistic map</u>:

$$x \ ? \ \ a * x * (1. - x) \qquad \text{[Equation 4]}$$

The population is at a fraction of capacity x at a given time. At the next reproductive cycle, the population is $b=a*(1-x)$ times as large. This is one of the simplest maps you can write for populations or any system, but it is turns out to have a behavior that, for some values of a, is quite complex.

<u>A plot of the logistic map (function)</u> and its close relatives looks something like this, where the height of the peak is $a/4$:



As you will see in homework problem #2 of set #4, the *types of behaviors* shown by this map, which vary with a, don't depend on the exact mathematical form, though the details vary (such as what values of a give stability.) The key feature that gives all of the intricate behavior is that there is a peak in the plot of the map function.

CHARACTERISTIC BEHAVIORS OF MAPS

The logistic map has a number of behaviors. Which behavior is seen depends on the value of the growth parameter a. Note that a must be less than or equal to 4 (otherwise, if $x=1/2$ one year, the next year's x, $a*x*(1-x)$, is greater than 1, exceeding environmental capacity.)

When the growth parameter $a < 1$, the next generation is always smaller than the current generation. The ratio of the new x to the old x is $a*(1-x)$, and since $(1-x) < 1$, the product of a and $(1-x)$ is less than 1, when $a < 1$. For this parameter, there are not enough offspring to fully replace the current generation. When the ecosystem is such that $a < 1$, the population goes to zero.

In lab, you tried $a=2.5$. In this case, you saw that the cone went to a fixed position x, with x? 0. This is the case of a stable, fixed population – the "system" "finds" a point where the births balance the deaths exactly. This is how you like your oven, cruise control, and thermostat to function: by self-adjusting to a fixed value.

What about $a=3.5$? The population *oscillated.* The cones indicating the size of the population hopped back and forth between two positions. One year, the population is a bit "high", so the population declines due to overcrowding, but the next year, there are more resources available, enough for the population to go back to its higher value. This is a *period-2* oscillation: the population repeats itself, or is periodic, with a period of 2. In this case, did you see the butterfly effect?

When $a=3.8$, you saw much more complex behavior. *Though the population in one year depends in a well-defined way on the previous year's population, the exact details of the population in future years becomes essentially unpredictable, due to the butterfly effect, or sensitivity to small changes.*

There are a number of other behaviors seen, for different a, such as *period-4, period-3, intermittency, chaotic bands, etc.,* that we might mention.