Lec5

- Nonlinear systems – chaos
- Phase space, Poincare maps, strange attractors
- Period doubling
- Lorenz model, balls in boxes ...

Real pendulum

Variables $\theta(t)$, $\omega(t)$ equation of motion:

$$\frac{d\omega}{dt} = -\frac{g}{l}\sin(\theta) - k\frac{d\theta}{dt} + F\sin(\omega_D t)$$
$$\frac{d\theta}{dt} = \omega$$

Integrate/solve by introducing discrete time $t = ndt, n = 1\ldots$ and find recurrence relations of form $\theta_{n+1} = \theta_n + \ldots$

Exactly analagous to 1D motion with $x \rightarrow \theta$, $p \rightarrow \omega$
Use same code!
(note: θ angle - restrict to $-\pi \rightarrow \pi$)

Simple observations

- Initially transients seen - remnant of decaying natural oscillation

- Small driving force, small amplitude, motion in step with driving force - like *hamrmonic* case

- Larger F − apparently random or *chaotic* behavior seen.

- Windows of regular motion found at larger F!

- *Cannot* be truly random - motion deterministic. Something more subtle happening ...

Sensitivity to initial conditions

Two identical pendula with *slightly* different initial conditions.

- In regular regime: motions *converge* with time

- In chaotic regime : diverge!

- In first case poor knowledge of initial conditions is *irrelevant* to predicting long time motion

- In other case implies *no predictability* at long times (eg. weather ...)

Phase space

Useful to examine motion not as (t, θ) and (t, ω) but in *phase space* (θ, ω).

- Regular (non-chaotic) motion yields *simple closed curve*.

- Chaotic motion – much structure. Many nearly closed orbits, sudden departures to new orbits, never repeating.

Poincare plots

Instead of plotting entire phase space trajectory, plot (θ, ω) *only* at multiples of time period of driving force.

- For regular motion - single point seen.

- For chaotic motion - non space filling structure seen. Does *not* depend on initial conditions

- Predictable aspect of chaotic motion – called a *strange attractor*. *All* chaotic motions of system approach a motion on the attractor.

- *Not* a 1D curve – in general *fractal* object - later.

Period doubling

- At $F = 1.35$ same period as F
- At $F = 1.44$ we see motion has *twice* period of driving force
- At $F = 1.465$ four times driving period $T = T_D$
- Continues. Successively smaller increases in F yield doublings of the period of the motion. $T = \infty$ at finite F!
- Period doubling route to chaos seen in many systems. Furthermore
$$\delta_n = \frac{F_n - F_{n-1}}{F_{n+1} - F_n} \lim_{n \to \infty} = \delta \sim 4.669..$$
Feigenbaum delta

Lorenz model

- Another example of model showing chaos.

- (Very)-simplified model of convective fluid flow – container containing fluid with bottom and top surfaces held at different temperatures.

- Three variables x, y, z corresponding to temperature, density and fluid velocity

- Three parameters σ, r, b (temperature difference and fluid parameters)

- Full solution involves *Navier-Stokes* and very many variables. Weather simulations etc.

Lorenz equations

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= -xz + rx - y \\
\frac{dz}{dt} &= xy - bz
\end{aligned}
$$

Discretize time and solve as before

Set $\sigma = 10.0$, $b = 8/3$. r measures temperature difference. Analogous to F in pendulum example.

$r = 5$ - settles to point - simple convective flow.

$r = 25$ - chaos – Lorenz attractor - chaotic or turbulent flow