Quantum Field Theory II Problem Set 1

Prof. C. Armendariz-Picon Due Sept. 11, 2006

Exercise 1: Statistics and Gaussian integrals

In this problem set we explore different properties of Gaussian integrals. They will be the foundation of our evaluation of path integrals in field theory. It turns out that in this formulation there is also a very close relationship with the properties of Gaussian random variables, which we shall also analyze.

a) Let D_{ij} be a symmetric positive-definite square matrix. Show that

$$\int \prod_i d\phi_i \exp\left(-\frac{1}{2}\sum_{i,j} D^{-1}_{ij}\phi_i\phi_j\right) = \det{}^{1/2}(2\pi D). \tag{1}$$

If we regard the ϕ_i as random variables, the previous equation implies that

$$P(\phi_i) \equiv \frac{1}{\det^{1/2}(2\pi D)} \exp\left(-\frac{1}{2}\sum_{i,j} D^{-1}_{ij}\phi_i\phi_j\right) \tag{2}$$

is a normalized probability distribution function, that is the probability to measure any value of the set of random variables is one,

$$\int \prod_i d\phi_i P(\phi_i) = 1. \tag{3}$$

b) Show that

$$\frac{1}{\det^{1/2}(2\pi D)} \int \prod_i d\phi_i \exp\left(-\frac{1}{2}\sum_{i,j} D^{-1}_{ij}\phi_i\phi_j + i\sum_i J^i\phi_i\right) = \exp\left(-\frac{1}{2}\sum_{i,j} J^i D_{ij} J^j\right), \tag{4}$$

and verify that, up to the determinant, the result is exactly what you would expect from the method of the steepest descent (evaluate the integrand at the extremum of the argument inside the exponential.)

c) Show that

$$I_{i_1\cdots i_n} \equiv \frac{1}{\det^{1/2}(2\pi D)} \int \left(\prod_i d\phi_i\right) \phi_{i_1}\cdots\phi_{i_n} \exp\left(-\frac{1}{2}\sum_{i,j} D^{-1}_{ij}\phi_i\phi_j\right) = \sum_{\text{pairings}} \prod_{\text{pairs}} D_{\text{paired i's}} \tag{5}$$

Hint: Use the result from part b). For example,

$$\begin{aligned} I_{i_1} &= 0 && (6)\\ I_{i_1 i_2} &= D_{i_1 i_2} && (7)\\ I_{i_1 i_2 i_3} &= 0 && (8)\\ I_{i_1 i_2 i_3 i_4} &= D_{i_1 i_2}D_{i_3 i_4} + D_{i_1 i_3}D_{i_2 i_4} + D_{i_1 i_4}D_{i_2 i_3} && (9) \end{aligned}$$

Note that equation (5) lends itself to a graphical interpretation, which leads to the Feynman rules we shall discuss in class. In the meantime, use the previous result to

d.) calculate

$$\frac{1}{\sqrt{2\pi}\sigma}\int d\phi\, \phi^{2n} \exp\left(-\frac{\phi^2}{2\sigma^2}\right). \tag{10}$$

Note that the integrals I are the are the moments of the distribution,

$$I_{i_1\cdots i_n} = \langle \phi_{i_1}\cdots\phi_{i_n}\rangle = \int \left(\prod d\phi_i\right) \phi_{i_1}\cdots\phi_{i_n} P(\phi_i) \tag{11}$$

Thus, equation (6) says that the Gaussian variables we have defined have zero mean, while equation (7) relates the covariance matrix and D,

$$\langle \phi_i \phi_j\rangle = D_{ij}. \tag{12}$$

Equation (8) simply means that Gaussian distributions have zero skewness.

Following along the same lines, we can identify the integral in equation (4) as the characteristic function of the distribution,

$$\Phi(J^i) = \langle \exp\left(i\sum_i J^i\phi_i\right)\rangle, \tag{13}$$

from which the different momenta follow by first differentiating with respect to J as many times as needed, and then setting $J^i = 0$. The logarithm of the characteristic function is the cumulant generating function,

$$K(J^i) = \log\Phi(J^i) = -\frac{1}{2}\sum_{i,j} J^i D^{-1}_{ij} J^j. \tag{14}$$

Hence, a zero-mean Gaussian distribution only has one non-vanishing cumulant, the coefficient of the term quadratic in J. As we shall later discuss the cumulant generating function is the equivalent of the generator of connected Feynman diagrams.

Sometimes we might not be interested in the value of a particular random variable, say, ϕ_1. We can then marginalize the distribution over that variable by "integrating it out":

$$P(\phi_2,\ldots,\phi_n) = \int d\phi_1 P(\phi_1,\phi_2,\ldots,\phi_n). \tag{15}$$

e.) Find the marginalized probability distribution $P(\phi_2,\ldots,\phi_n)$ in the case under discussion (Gaussian random variables with zero mean).

Exercise 2: From the Hamiltonian to the Lagrangian path integral

Assume that the Hamiltonian is quadratic in $\vec{\pi}$, and that the coefficients of such quadratic terms do not depend on $\vec{\phi}$. Using the results of Exercise 1, show that

$$\int D\phi D\pi \exp\left[i\int(\vec{\pi}\cdot\dot{\vec{\phi}} - H(\vec{\phi},\vec{\pi},t))dt\right] = \mathcal{N}\int D\phi \exp\left[i\int L(\vec{\phi},\dot{\vec{\phi}},t)dt\right], \qquad (16)$$

where $\mathcal{N}$ is a constant.

Exercise 3: Matrix elements of time-ordered products as path integral

Show that

$$\int D\phi\, \phi_{a_1}(t_1)\cdots\phi_{a_n}(t_n)\exp\left(i\int dt L\right) = \langle\vec{\phi}_f, t_f|T\left(\phi_{a_1}(t_1)\cdots\phi_{a_n}(t_n)\right)|\vec{\phi}_i, t_i\rangle, \qquad (17)$$

where

$$|\vec{\phi}_i, t_i\rangle = U^{-1}(t_i, t_0)|\vec{\phi}_i\rangle, \qquad (18)$$

$$|\vec{\phi}_f, t_f\rangle = U^{-1}(t_f, t_0)|\vec{\phi}_f\rangle, \qquad (19)$$

and verify that the latter are the eigenvectors of the Heisenberg field operators $\vec{\phi}(t)$.