Quantum Field Theory II

Problem Set 12

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Exercise 33: Weyl Spinor Lagrangian

Verify that the Lagrangian

$$\mathcal{L} = -i\bar{\phi}\bar{\sigma}^\mu \partial_\mu \phi - \frac{1}{2} m\phi\phi - \frac{1}{2} m\bar{\phi}\bar{\phi} \tag{1}$$

leads to a real action.

Exercise 34: Weyl spinors

Verify that the Dirac Lagrangian

$$\mathcal{L} = -\bar{\Psi}\gamma^\mu \partial_\mu \psi - m\bar{\Psi}\Psi, \tag{2}$$

where $\bar{\Psi} = \Psi^\dagger \beta$ and

$$\Psi = \begin{pmatrix} \epsilon\phi^* \\ \chi \end{pmatrix}, \tag{3}$$

yields two copies of the Lagrangian (1). Check that replacing $\Psi \to \Psi_L = \frac{1-\gamma_5}{2}\Psi$ leads to the Lagrangian of a massless Weyl spinor. What is the Lagrangian if we replace $\Psi \to \Psi_R = \frac{1+\gamma_5}{2}\Psi$.

Exercise 35: Majorana Spinor

Show that

$$\Psi_M = \begin{pmatrix} \epsilon\chi^* \\ \chi \end{pmatrix} \tag{4}$$

is indeed a Majorana spinor, $\Psi_M = \beta C \Psi_M^*$. Verify that its Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2}\Psi^T C\gamma^\mu \partial_\mu \Psi + \frac{1}{2}\Psi^T C\Psi. \tag{5}$$

Exercise 35: Field eigenstates

Show that

$$|\vec{\psi}\rangle = \exp\left(-i\sum_a \hat{\pi}_a \psi_a\right)|0\rangle \tag{6}$$

and

$$\langle\vec{\psi}| = \langle 0| \left(\Pi_a \hat{\psi}_a\right) \exp\left(i\sum_a \hat{\pi}_a \psi_a\right) \tag{7}$$

are left and right eigenvectors of the operators $\hat{\psi}_a$ with eigenvalues ψ_a. Check that

$$\langle\vec{\psi}'|\psi\rangle = \Pi_a(\psi_a - \psi'_a). \tag{8}$$

Exercise 36: Fermionic path integrals

Show that

$$\int D\psi D\pi \, \psi_{a_1}\pi_{b_1}\cdots\psi_{a_n}\pi_{b_n} \exp\left(-i\sum_{ab} D^{-1}_{ab}\pi^a\psi^b\right) = \sum_{\text{pairings}} \delta\cdot\Pi_{\text{pairs}}(-iD)_{\text{paired a and b}}. \quad (9)$$

In the equation above, δ is the sign of the permutation needed to permute the fields on the left hand side of the equation into the set of ordered pairs on the right hand side.

Exercise 37: Elastic scattering of fermions and antifermions

Calculate the elastic scattering amplitude of *i*) two anti-fermions and *ii*) a fermion anti-fermion pair to order λ^2 in Yukawa's theory.

Exercise 38: Holiday gift

The Wess-Zumino model is the first supersymmetric model in four dimensions discussed in the western literature.[1] Its Lagrangian is

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{1}{2}m^2(A^2+B^2) + \frac{1}{2}\Psi^T C\gamma^\mu\partial_\mu\Psi + \frac{1}{2}m\Psi^T C\Psi + \\ +\lambda A\Psi^T C\Psi + i\lambda B(\Psi^T C\gamma_5\Psi) - m\lambda A(A^2+B^2) - \frac{\lambda^2}{2}(A^2+B^2)^2,\end{aligned} \quad (10)$$

where A is a scalar, B a pseudoscalar, and Ψ is a Majorana spinor.

1. Derive the Feynman rules for this theory
2. Calculate the self-energy of the two scalars to order λ^2.
3. Do you see anything remarkable?

[1] J. Wess and B. Zumino, "Supergauge Transformations in Four-Dimensions," Nucl. Phys. B **70**, 39 (1974).