

This final exam consists of five problems, for a total of 100 points. I have arranged the problems in order of increasing difficulty. Make sure that you attempt to answer all the questions and all the problems. If you get stuck with a problem for too long, proceed to the next one. Please answer as clearly, precisely and thoroughly as possible. May the muses be with you.

Problem 1 (20 points)

The Hamiltonian of the harmonic oscillator is

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right).$$

1. What relation defines the vacuum $|0\rangle$ of the oscillator?
2. What is the meaning of the factor of $\frac{\hbar\omega}{2}$?
3. Let $|n\rangle$ be an eigenvector of $\hat{a}^\dagger \hat{a}$. Express $|n\rangle$ in terms of the vacuum (you do not have to worry about normalization factors.)
4. What can you say about $\hat{a}|n\rangle$ and $\hat{a}^\dagger|n\rangle$?

Problem 2 (20 points)

Consider an operator \hat{A} which in the Schrödinger representation does not depend on time.

1. How is the operator $\hat{A}(t)$ in the Heisenberg representation related to the operator \hat{A} in the Schrödinger representation? Do not assume that the Hamiltonian is time-independent.
2. Show that the operator $\hat{A}(t)$ in the Heisenberg representation obeys the equation

$$i\hbar \frac{d\hat{A}}{dt} = [\hat{A}(t), \hat{H}(t)],$$

where \hat{H} is the eventually time-dependent Hamiltonian of the system.

Problem 3 (20 points)

Let \hat{x} and \hat{p} be two operators satisfying $[\hat{x}, \hat{p}] = i\hbar$, and let $|x\rangle$ and $|p\rangle$ denote their corresponding eigenvectors, normalized such that $\langle x|x'\rangle = \delta(x - x')$ and $\langle p|p'\rangle = \delta(p - p')$. Show that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar}px\right).$$

Make sure that you only use the information given above.

Problem 4 (20 points)

Consider a particle scattering off a potential

$$V(x) = V_0 \delta(x).$$

Compute the reflection and transmission coefficients Q and S . (*Hint:* Because the potential is unbounded, the derivative of the wave function *does not* have to be continuous.)

Problem 5 (20 points)

We let a beam of spin 1/2 particles traverse a Stern-Gerlach apparatus, and select those particles that were deflected upwards (along the plus z direction.) Subsequently, at time t_0 , the (selected) beam enters an apparatus that contains a constant, homogeneous magnetic field along the plus x direction. The Hamiltonian of the system inside this apparatus is

$$H = \frac{1}{2} \vec{S} \cdot \vec{B},$$

where \vec{S} is the spin operator and \vec{B} the constant magnetic field in the plus x direction. Finally, at time t_1 , we decide to measure the spin of the particles along the z direction.

1. What is the state of the system at time t_0 ?
2. What is the state of the system at time t , where $t_0 < t < t_1$?
3. What is the state of the system at time $t > t_1$?
4. What is the probability to find the spin of the particles along the plus z direction in the measurement at time t_1 ?

You might need the following relations:

$$\begin{aligned} S_z |\pm z\rangle &= \pm |\pm z\rangle, \\ S_x |\pm x\rangle &= \pm |\pm x\rangle, \\ |\pm z\rangle &= \frac{1}{\sqrt{2}} (|+x\rangle \pm |-x\rangle). \end{aligned}$$