Quantum Mechanics I Problem Set 4

Prof. C. Armendariz-Picon Due Oct 6, 2006

Exercise 26

Let ρ be the density matrix of a system. Show that a state is pure if and only if $\rho = \rho^2$.

Exercise 27

Let ρ be again the density matrix of a system. Sow that

i) $\rho^\dagger = \rho$

ii) $tr\rho = 1$.

Exercise 28

Show that the expectation value of an observable A is given by

$$\langle A \rangle = tr(\rho A),$$

and show that the probability of finding the eigenvalue a upon measurement of A is

$$p_a = tr(\rho P_a),$$

where P_a is the operator that projects onto the eigenspace of a.

Please turn for more [...] exercises.

Problem 29

Consider a two-level system. Let $|1\rangle$ and $|2\rangle$ be an orthonormal basis of eigenvectors of the observable N in the Hilbert space, and assume that an observable A has eigenvectors and eigenvalues

$$|a\rangle = \alpha|1\rangle + \beta|2\rangle, \quad A|a\rangle = a|a\rangle,$$
$$|b\rangle = -\beta^*|1\rangle + \alpha^*|2\rangle, \quad A|b\rangle = b|b\rangle.$$

Suppose that the initial state of the system is

$$\rho = |\psi\rangle\langle\psi|, \quad \text{where} \quad |\psi\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle.$$

1. What is the probability of measuring a? Derive your answer both using the postulates of QM as presented in class, and the density matrix formalism.
2. Imagine that we perform an intermediate measurement of the observable N. What is the probability of measuring a after measurement of N? Derive your answer using both the postulates of QM and the density matrix formalism. (Ignore here and in the following the time evolution between measurements.)
3. Imagine that after performing the measurement of N we *select* the systems in state $|1\rangle$. What is the probability then of subsequently finding a? Derive your answer both using the postulates of QM and the density matrix formalism.
4. Compare the answers to the previous questions. Analyze the type of states you have after each measurement (pure or mixed). To what extent are pure and mixed states similar/different?

Problem 30

Show that the operator $U(t, t_0) = \exp\left[-\frac{i}{\hbar} H \cdot (t - t_0)\right]$ is unitary ($U(t, t_0)$ is called the time propagator.)

Problem 31

Show that the propagator satisfies the Schrödinger equation,

$$i\hbar \frac{dU(t, t_0)}{dt} = H(t)U(t, t_0).$$