Cramer's Rule for solving systems of equations

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Systems of Equations

Suppose we wish to solve the following system of two equations with two unknowns:

$$\begin{aligned} x + 2y &= 6 \\ 3x + 4y &= 8 \end{aligned}$$

"To solve this system" means to find all pairs (x, y) that satisfy all of the equations.

<u>method 1: Brute-Force Algebra</u>
The first technique one learns is to do this:

1. Solve one equation for one of the unknowns. If, for some reason, we are asked to determine values for x then for y, we should solve for y first.

   Let's work with the first equation and solve for y:

   $$y = \frac{6 - x}{2}$$

2. Using this expression in the second equation,

   $$3x + 4\left(\frac{6 - x}{2}\right) = 8$$

   we obtain an equation for one unknown x. Solving for x, we find

   $$x = -4.$$

3. Using this value for x in the expression for y, we obtain

   $$y = \frac{6 - (-4)}{2} = 5.$$

So, $x = -4$ and $y = 5$.
Let's check this:

$$\begin{aligned} (-4) + 2(5) &= 6 \qquad \surd \\ 3(-4) + 4(5) &= 8 \qquad \surd \end{aligned}$$

method 2: Geometrical Interpretation

Geometrically,

$$x + 2y = 6$$

is the equation of a line. (Maybe it's more recognizable to you if you solve this equation for y, as you did above: $y = \frac{6-x}{2}$. Rewrite this as $y = -\frac{1}{2}x + 3$. When any pair (x, y) satisfies this equation, the point with coordinates (x, y) is on this line. For instance, each of these pairs is on this line:

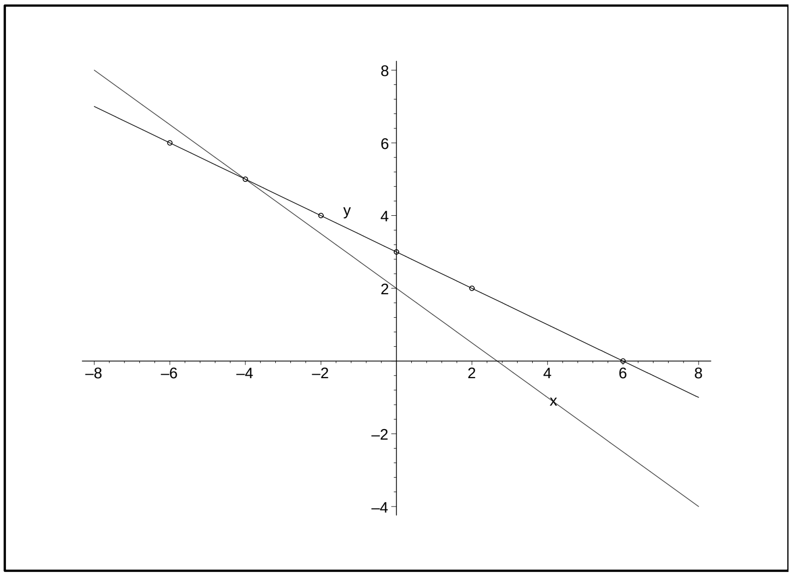
$$(-6, 6) \qquad (-4, 5) \qquad (-2, 4) \qquad (0, 3) \qquad (2, 2) \qquad (6, 0)$$

Similarly,

$$3x + 4y = 8$$

is the equation of another line. Generally, a different set of points (x, y) will lie on this line:

$$(-8, 8) \qquad (-4, 5) \qquad (0, 2) \qquad (4, -1) \qquad (8, -4)$$



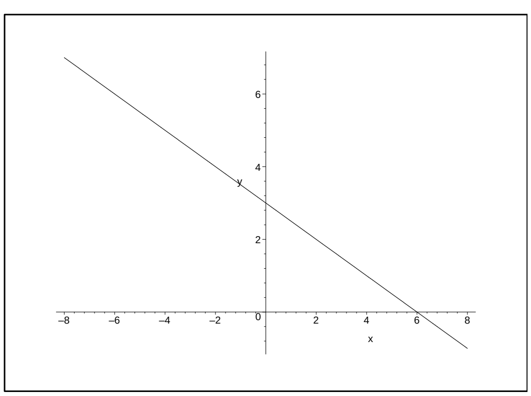
Observe that point $(-4, 5)$ lies on both lines. This is the solution of the system of equations. How can this point $(-4, 5)$ be characterized? It is the intersection of the two lines. Thus, **the solution of the system of equations is the common interesection of all of the lines**.

Observe that the solution to the above system is unique. Note, however, that there are situations when the solutions are not unique.

For instance, the system

$$x + 2y = 6$$
$$2x + 4y = 12$$

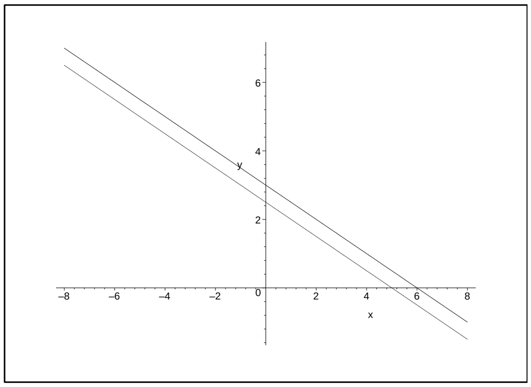
has infinitely many solutions. Why? It is because the two lines coincide. Note that the second equation is twice the first equation.



Next consider the system

$$\begin{aligned} x + 2y &= 6 \\ 2x + 4y &= 10 \end{aligned}$$

has no solutions. Why? It is because the two lines are distinct but parallel. Note that the second equation can be re-expressed as $x + 2y = 5$. Since $x + 2y$ can't equal both 6 and 5 simultaneously, no point (x, y) can lie on both lines.



Clearly, the solution of a system of equations is unique only when the lines have different slopes.

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method 3: Cramer's Rule with matrices and determinants

Cramer's Rule is an algorithm to solve a system of n equations with n unknowns using "matrix methods". In some sense, it takes care of all of the algebra for you and reduces the problem to an arithmetic problem.

Suppose we wish to solve the following system of equations:

$$\begin{aligned} Ax + By &= P \\ Cx + Dy &= Q \end{aligned}$$

where A, B, C, D, P, and Q are constants. Note the alignment of terms. Let us call this the "standard form" of the system of equations.

Once in standard form, we can write the system using a matrix as follows:

$$\underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\text{coefficient matrix}} \quad \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{solution column vector}} \quad = \quad \underbrace{\begin{pmatrix} P \\ Q \end{pmatrix}}_{\text{constant column vector}} \quad .$$

The left-hand side is the "[matrix] multiplication" of a matrix and a column vector.

To help you visualize "matrix multiplication" let us write:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}$$

as

$$\begin{pmatrix} \overrightarrow{A \quad B} \\ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\Big\downarrow = \begin{pmatrix} P \\ \end{pmatrix} \qquad\qquad Ax + By = P$$

and

$$\begin{pmatrix} \\ \overrightarrow{C \quad D} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\Big\downarrow = \begin{pmatrix} \\ Q \end{pmatrix} \qquad\qquad Cx + Dy = Q$$

In order to apply Cramer's rule to the system of equations

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix}$$

we have to compute the **determinant** of the coefficient matrix, which is written as

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}.$$

The determinant is evaluated by expanding along a row (say, the AB-row) and forming terms with the element and a certain subdeterminant (called its *minor*) and sign-factor. Each element's minor is visualized by ignoring the elements in the same row and in the same column. For example, the minor of A is D, and the minor of B is C. The sign-factor follows this checkerboard pattern:

$$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

So,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC$$

It is instructive to recall that the cross-product can be computed using a (3×3)-determinant:

$$\begin{aligned} \vec{A} \times \vec{B} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \hat{\imath} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} + \hat{k} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \hat{\imath} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \\ &= \hat{\imath}\,(A_yB_z - A_zB_y) - \hat{\jmath}\,(A_xB_z - A_zB_x) + \hat{k}\,(A_xB_y - A_yB_z) \end{aligned}$$

where the subdeterminants have been evaluated.

Now, we're finally ready for Cramer's Rule.

In order to solve the system of equations

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} P \\ Q \end{pmatrix},$$

calculate the following ratios of determinants

$$x = \frac{\begin{vmatrix} P & B \\ Q & D \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} A & P \\ C & Q \end{vmatrix}}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}}$$

Observe that, for the *first* variable x, the determinant $\begin{vmatrix} P & B \\ Q & D \end{vmatrix}$ is gotten by taking $\begin{vmatrix} A & B \\ C & D \end{vmatrix}$ and replacing the *first* column with the constant column vector. Similarly, for the *second* variable y, the determinant $\begin{vmatrix} A & P \\ C & Q \end{vmatrix}$ is gotten by replacing the *second* column with the constant column vector.

Cramer's Rule generalizes in the obvious way to larger systems of equations.

Let us try our numerical example

$$x + 2y = 6$$
$$3x + 4y = 8.$$

The matrix form is

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

Now, we calculate the following ratios of determinants

$$x = \frac{\begin{vmatrix} 6 & 2 \\ 8 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{(6)(4) - (2)(8)}{(1)(4) - (2)(3)} = \frac{8}{-2} = -4$$

$$y = \frac{\begin{vmatrix} 1 & 6 \\ 3 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{(1)(8) - (6)(3)}{(1)(4) - (2)(3)} = \frac{-10}{-2} = 5$$

You might wonder about the case when the determinant of the coefficient matrix [which appears in the denominator] is zero. That determinant is zero when rows are proportional to each other since $\begin{vmatrix} A & B \\ kA & kB \end{vmatrix} = (A)(kB) - (B)(kA) = 0$. For example, the system with infinitely many solutions

$$x + 2y = 6$$
$$2x + 4y = 12$$

has as its determinant of coefficients $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1)(4) - (2)(2) = 0$. The system with no solutions also has zero determinant.